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Two Point Boundary Problems in Banach Space

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1. INTRODUCTION

Let X be a Banach space, $[X]$ the Banach algebra of bounded linear operators on X into X. The continuity, derivative and integral of a function $y(t)$, or $Y(t)$, from the real interval $t \geqslant 0$ into X, or $[X]$, is defined in terms of the norm topology. These concepts together with the necessary calculus may be found in [*1*], pp 58–123. It is assumed throughout that A and B are continuous mappings from $t \geqslant 0$ into $[X]$.

We consider below the questions of existence and uniqueness of solutions to

$$y''(t) = A(t)y'(t) + B(t)y(t) \tag{1}$$

satisfying the boundary conditions

$$y(0) = \alpha, \qquad y(\tau) = \beta \qquad \alpha, \beta \in X \tag{2}$$

where y is of class C^2 on the compact interval $[0, \tau]$ into X. Our main objective is to impose conditions on the operator functions A, B and the interval length $\tau > 0$ so that (1), (2) has a unique solution for every choice of boundary values α, β.

Applying an operational calculus for commutative normed rings, we give a necessary and sufficient condition for unique sulvability when A and B are constant and commute. In general, however, we can only state quantitative sufficient conditions (i.e. inequalities involving norms), some of which generalize classical results for the scalar case. When X is Hilbert space, several qualitative conditions on A and B are given which make (1), (2) uniquely solvable.

If X has finite dimension, solutions to (1), (2) are unique if and only if (1), (2) is solvable for all $\alpha, \beta \in X$. ([*2*], p. 387) Consequently, a study of the boundary value problem reduces to determining conditions on the $n \times n$ matrices A, B and the interval length $\tau > 0$ under which the only solution to (1) vanishing at $t = 0$, τ is the trivial solution; i.e. disconjugacy or non-

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oscillation conditions. Hartman and Wintner have given disconjugacy criteria in [3] when (1) is non-self-adjoint. For the self-adjoint case, sufficient conditions for disconjugacy have been given by Barrett [4] and Reid [5]. The related questions of oscillation or non-oscillation for large t for the system (1) and/or the corresponding matrix system are considered in [4]-[10].

2. Preliminaries

Let Θ denote the zero of either X or $[X]$ and let I be the identity in $[X]$. Let U and V be the solutions to the operator equation

$$Y'' = AY' + BY \tag{3}$$

satisfying the initial conditions $U(0) = \Theta$, $U'(0) = I$ and $V(0) = I$, $V'(0) = \Theta$. Since solutions to initial value problems for either (1) or (3) exist uniquely on any interval where A and B are continuous, the solution to (1) satisfying the initial conditions $y(0) = \alpha$, $y'(0) = \gamma$ is given by

$$y(t) = U(t)\gamma + V(t)\alpha. \tag{4}$$

Consequently, we have

Lemma 1. *On a compact interval* $[0, \tau]$, *solutions to* (1), (2) *are unique if, and only if,* $U(\tau)$ *is one-one and* (1), (2) *is solvable for all* α, β *if, and only if,* $U(\tau)$ *is onto.*

Lemma 2. *Let* A *and* B *be continuous on* $[0, \tau]$ *into* $[X]_c$, *the ideal in* $[X]$ *of completely continuous operators. Then* $U(\tau)$ *is one-one if, and only if,* $U(\tau)$ *is onto.*

Proof. Double integration of (3) together with the initial conditions yields $U(\tau) = \tau I + \int_0^\tau (\tau - s)[AU' + BU]\, ds$. If L denotes the integral term on the right, the conditions on A and B insure that $L \in [X]_c$. Using the terminology in [11], let λ and μ denote the dimensions of the null space and the defect subspace of $\tau I + L$. By Theorem 3.1 of [11], the index $\mu - \lambda$ of $\tau I + L$ equals the index of τI when L is completely continuous. But the index of the invertible operator τI is zero so $\mu = \lambda$ and $U(\tau)$ is one-one ($\lambda = 0$) if, and only if, $U(\tau)$ is onto ($\mu = 0$).

We note that if the boundary conditions (2) are replaced by $y'(0) = \alpha$, $y(\tau) = \beta$ we can replace U by V in Lemma's 1 and 2. A similar correspondence exists between the remaining possible boundary conditons and the operators U' and V'.

The hypothesis of complete continuity in Lemma 2 cannot be dropped. For let $X = l_2$ and for $n \geqslant 1$ let $e_n = (0,..., 0, 1, 0,...)$ denote the usual complete orthonormal set. Define the constant operator B by

$$Bx = \sum x_n Be_n = -\sum x_n[n\pi/(n+1)]^2 e_n \text{ for } x = \sum x_n e_n \in X.$$

B is linear, $\| B \| = \pi^2$ and the solution U to (3) is given in terms of the complete orthonormal set by

$$U(t)e_n = [(n+1)/n\pi] \sin [tn\pi/(n+1)]e_n \, .$$

Then $U(1)$ is one–one but $\Sigma (1/n)e_n$ does not belong to its range so $U(1)$ is not onto.

Lemma 3. *If C is continuous on $[0, \tau]$ into $[X]$ and $Z(t)$ is a solution to $Z' = CZ$ in $[X]$ with $Z(O)$ invertible, then $Z(t)$ is invertible on $[0, \tau]$.*

Proof. On some interval to the right of zero, $Z^{-1}(t)$ exists and is a solution of $W' = -WC$. Applying Gronwall's inequality to $\| W(t)\| \leqslant \| W(0)\| + \int_0^t \| W(s)\| \, \| C(s)\| \, ds$ shows that $Z^{-1}(t)$ remains bounded on any subinterval of $[0, \tau]$ where it exists. Since the invertible operators form an open set in $[X]$, Z^{-1} must exist on $[0, \tau]$.

Lemma 4. *For A continuously differentiable and B continuous on $[0, \tau]$ into $[X]$, equation (1) is equivalent to the system $z'' = Cz$ under the substitution $y(t) = R(t) \, z(t)$ where $R(t)$ is the solution in $[X]$ of $R' = \frac{1}{2}AR$, $R(0) = I$ and $C(t) = R^{-1}(t)[\frac{1}{4}A^2(t) + B(t) - \frac{1}{2}A'(t)] \, R(t)$. If A and B are constant and commute, $C = \frac{1}{4}A^2 + B$.*

Proof. The first statement is a straight forward calculation. The last statement is a consequence of $R(t) = \exp(\frac{1}{2}tA)$ which commutes with A^2 and B when A and B commute.

3. Constant Coefficients

Assume $A, B \in [X]$ are independent of $t \geqslant 0$ and $AB = BA$. Then the solution U to (3) is given by

$$U(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^{n+k+1} \, n!}{(n+k+1)! \, (n-k)! \, k!} A^{n-k}B^k. \tag{5}$$

Moreover, $V(t) = U'(t) - AU(t)$ so that every solution of (1) can be written in terms of the above series using (4).

Define the complex valued function $u(t, \lambda, \mu)$ for t real, λ and μ complex numbers, by

$$u(t, \lambda, \mu) = \begin{cases} te^{\lambda(t/2)} & \lambda^2 + 4\mu = 0 \\ 2e^{\lambda(t/2)}(\lambda^2 + 4\mu)^{-1/2} \sinh(t/2)(\lambda^2 + 4\mu)^{1/2} & \lambda^2 + 4\mu \neq 0 \end{cases} \tag{5'}$$

Then $u(t, \lambda, \mu)$ is the solution to the scalar equation

$$y'' = \lambda y' + \mu y \qquad y(0) = 0, \qquad y'(0) = 1$$

and has the expansion of (5) with λ, μ replacing A and B. The function u plays an important role in the following theorem.

THEOREM 1. *Let X be a complex Banach space and assume $A, B \in [X]$ commute. Then the BVP* (1), (2) *is uniquely solvable on* $[0, \tau]$ *if, and only if,* $\tau(\lambda^2 + 4\mu)^{1/2} \neq 2n\pi i$ *for all integers* $n \neq 0$ *and all pairs* (λ, μ) *belonging to the joint spectrum* S of A, B.

Proof. Let $C\{A, B\}$ denote the commutative subalgebra of $[X]$ generated by A and B. Let R be the ring of analytic complex valued functions $f(\lambda, \mu)$ defined on S and assume the non-equality holds. Then the function $u(\tau, \lambda, \mu)$ defined in (5′) belongs to R and does not vanish on the compact set S. Since A and B are generators of $C\{A, B\}$, there is a homomorphism of R into $C\{A, B\}$ such that the functions $f(\lambda, \mu) = 1$, $g(\lambda, \mu) = \lambda$, $h(\lambda, \mu) = \mu$ in R correspond to I, A and B respectively in $C\{A, B\}$. ([*12*], pp. 91–94) From (5) and (5′) we see that $U(\tau)$ corresponds to $u(\tau, \lambda, \mu)$ and since $u^{-1}(\tau, \lambda, \mu)$ belongs to R, the corresponding element in $C\{A, B\}$ must be $U^{-1}(\tau)$. It then follows from Lemma 1 that (1), (2) is uniquely solvable on $[0, \tau]$.

Suppose, conversely, that $\tau(\lambda^2 + 4_\mu)^{1/2} = 2ni\pi$ for some $n \neq 0$, $(\lambda, \mu) \in S$. Then $u(\tau, \lambda, \mu)$ has a zero in S. But the spectrum of $U(\tau)$ is precisely the image of S under the mapping u, i.e. $\sigma(U(\tau)) = \{u(\tau, \lambda, \mu) : (\lambda, \mu) \in S\}$. ([*12*], p. 279) Consequently $0 \in \sigma(U(\tau))$ and $U(\tau)$ is not invertible so that either solutions to (1), (2) are not unique or do not exist for certain boundary values.

THEOREM 2. *If X is any Banach space and $A, B \in [X]$ are constant and commute with* $\| \frac{1}{4}A^2 + B \| \tau^2 < \pi^2$, *the* BVP (1), (2) *is uniquely solvable on* $[0, \tau]$.

Proof. According to Lemma's 1 and 4 it sufficies to show that $U(\tau)$ is invertible where U is the solution in $[X]$ to $Z''(t) = CZ(t)$, $Z(0) = \Theta$, $Z'(0) = I$ for $C = \frac{1}{4}A^2 + B$.

If X is a complex space, the result follows from Theorem 1. For replacing A and B by Θ and C we need $\tau(4\mu)^{1/2} \neq 2n\pi i$ for $\mu \in \sigma(C)$, $n \neq 0$. But $\sigma(C) \subset \{\mu : |\mu| \leqslant \|C\|\} \subset \{\mu : |\mu| < (\pi/\tau)^2\}$ where the last inclusion follows from the hypothesis. Thus the non-equality holds for $n \neq 0$.

If X is not complex, $X^+ = X \times X$ with component wise addition and scalar multiplication defined by $(a + ib)(x, y) = (ax - by, bx + ay)$ is a complex Banach space with $\|(x, y)\|^+ = \max\{\|x\|, \|y\|\}$. For $C \in [X]$, $C^+(x, y) = (Cx, Cy) \in [X^+]$ is the extension of C. Since $\|C\| = \|C^+\|^+$, we can apply the argument of the above paragraph to X^+ and C^+ and assert that $U^+(\tau)$ is invertible. It follows that $U(\tau)$ is invertible on X.

The choice $A = \Theta$, $B = -I$ (so that $U(t) = \sin t I$) shows that π^2 cannot be replaced in the above inequality by any smaller constant.

The constant coefficient case with noncommuting A and B will be postponed until the next section. One expects that an inequality rather similar to the one given in Theorem 2 would be sufficient for (1), (2) to be uniquely solvable. However, the technique of reducing equation (1) to an equivalent system $z'' = Cz$ without the y' terms requires a non-constant operator $C(t) = \frac{1}{4}A^2 + R^{-1}(t)\, BR(t)$ and Theorem 1 is no longer applicable.

4. Variable Coefficients

The next two theorems are obtained by considering a Fredholm integral equation equivalent to (1), (2). Conditions are put on A, B and $\tau > 0$ so that the integral operator is a contraction mapping on an appropriate function space.

Theorem 3. *Let $A(t)$ be continuously differentiable and $B(t)$ continuous on $[0, \tau]$ into $[X]$ and let $C(t)$ be as in Lemma 4. If $\int_0^\tau s(\tau - s) \|C(s)\|\, ds < \tau$, the* BVP (1), (2) *is uniquely solvable.*

Proof. Let F denote the Banach space of continuous X-valued functions $u(t)$ defined on $[0, \tau]$ with $\|u\|_F = \sup\{\|u(t)\| : 0 \leqslant t \leqslant \tau\}$. For fixed $\alpha, \beta \in X$ define the mapping L on F into F by $L(u) = w$ where

$$w(t) = \int_0^\tau G(t, s)\, C(s)\, u(s)\, ds + \frac{t}{\tau}(\beta - \alpha) + \alpha$$

and where

$$G(t, s) = \begin{cases} s(t - \tau)/\tau & 0 \leqslant s \leqslant t \\ t(s - \tau)/\tau & t \leqslant s \leqslant \tau \end{cases}$$

is the Green's function for the scalar equation $y'' = 0$. Let $\lambda = \tau^{-1} \int_0^\tau s(\tau - s) \| C(s) \| \, ds < 1$. Then for $u, v \in F$

$$\| L(u - v)(t) \| \leqslant \int_0^\tau | G(t, s) | \, \| C(s) \| \, ds \, \| u - v \|_F \leqslant \lambda \, \| u - v \|_F$$

so $\| (L(u - v) \|_F \leqslant \lambda \| u - v \|_F$ and L is a contraction. The unique fixed point of L is the (unique) solution to $z'' = Cz$, $z(0) = \alpha$, $z(\tau) = \beta$. Since L is a contraction independent of the choice of α, β we have that the BVP for the reduced equation is uniquely solvable on $[0, \tau]$. By Lemma 4 the same is true for (1), (2).

Corollary 1. *If A is continuously differentiable and B is continuous on $[0, \tau]$ into a commutative subalgebra of $[X]$ with $\int_0^\tau s(\tau - s) \| \tfrac{1}{4}A^2(s) + B(s) - \tfrac{1}{2}A'(s) \| \, ds < \tau$, the* BVP (1), (2) *is uniquely solvable.*

Proof. Under the present hypotheses the operator $R(t)$ in Lemma 4 commutes with A^2, B and A' on $[0, \tau]$ so that $C = \tfrac{1}{4}A^2 + B - \tfrac{1}{2}A'$ in Theorem 3.

As a special case of this corollary with $A = \Theta$ and X the real numbers we have; If a valued solution $u(t) \neq 0$ of $u'' = bu$ has two zeros in $[0, \tau]$ then $\int_0^\tau s(\tau - s) \, | b(s) | \, ds \geqslant \tau$. (cf Theorem 5.1, p. 345 of [*2*]).

Corollary 2. *If B is continuous on $[0, \tau]$ into $[X]$ and satisfies any of*

(i) $\tau \int_0^\tau \| B(t) \| \, dt \leqslant 4$

(ii) $\tau^2 \sup\{ \| B(t) \| : 0 \leqslant t \leqslant \tau \} < 8$

(iii) $\tau^3 \int_0^\tau \| B(t) \|^2 \, dt < 48$

then (1), (2) *with $A = \Theta$ is uniquely solvable.*

Proof. When A $= \Theta$ we have $C = B$ in Theorem 3. From the inequalities $s(\tau - s) \leqslant \tau^2/4$, $\int_0^\tau | G(t, s) | \, ds \leqslant \tau^2/8$ and $\int_0^\tau | G(t, s) |^2 \, ds \leqslant \tau^3/48$ we $\int_0^\tau | G(t, s) | \, \| B(s) \| \, ds < 1$ using (i), (ii), and (iii) respectively, where the Schwartz inequality is used for (iii). In each case the mapping of Theorem 3 is a contraction.

A restatement of condition (i) gives Lyapunov's result on the zeros of solutions to (1).

Corollary 3. [Lyapunov] *A necessary condition for the system $y'' = By$ to have a nontrivial solution with two zeros on $[0, \tau]$ is that $\int_0^\tau \| B(t) \| \, dt > 4/\tau$.*

When A and B are constant but $AB \neq BA$, we can replace $B(t)$ in Corollary 2 (or Theorem 3) by $C(t) = \tfrac{1}{4}A^2 + \exp((-t/2)A) \, B \exp((t/2)A)$

to obtain sufficient conditions for unique solvability. Weaker but more easily verified sufficient conditions are obtained by using the estimate $\| C(t)\| \leqslant \| \frac{1}{4}A^2 + B \| e^{t\|A\|}$ which follows by taking norms through the equation $C(t) = [\frac{1}{4}A^2 + B] + \frac{1}{2}\int_0^\tau [C(s)A - AC(s)]\, ds$ and applying Gronwalls inequality. Using this estimate, which is clearly valid when $AB = BA$, in Corollary 2 we get

Corollary 4. *If $A \neq \Theta$ and B are constant, then each of the following is sufficient that* (1), (2) *be uniquely solvable.*

(i) $\tau \| \frac{1}{4}A^2 + B \| (e^{\tau\|A\|} - 1) \leqslant 4 \| A \|$

(ii) $\tau^2 \| \frac{1}{4}A^2 + B \| e^{\tau\|A\|} < 8$

(iii) $\tau^3 \| \frac{1}{4}A^2 + B \| (e^{2\tau\|A\|} - 1) < 96 \| A \|$

Theorem 4. *Let $h(s) = \| A(s)\| + s \| B(s)\|$ and assume that* $\max\{\int_0^\tau sh(s)\, ds, \int_0^\tau (\tau - s)\, h(s)\, ds\} < \tau$. *Then* (1), (2) *is uniquely solvable on* $[0, \tau]$.

Proof. Let F denote the class of continuously differentiable X valued functions u defined on $[0, \tau]$ satisfying $u(0) = \Theta$. Then F with norm defined by $\| u \|_F = \sup\{\| u'(t)\| : 0 \leqslant t \leqslant \tau\}$ is a Banach space. Define the mapping L on F into F by $L(u) = w$ where

$$w(t) = \int_0^\tau G(t, s)[Au' + Bu]\, ds + \frac{t}{\tau}\beta$$

for $\beta \in X$ fixed. If the inequality of the theorem holds,

$$\lambda = \sup \left\{ \int_0^\tau | G_t(t, s)|\, h(s)\, ds : 0 \leqslant t \leqslant \tau \right\} < 1.$$

Since

$$\begin{aligned} \| L(u)'(t) - L(v)'(t)\| &= \left\| \int_0^\tau G_t(t, s)\Big[A(u' - v') \right. \\ &\qquad \left. + B \int_0^s (u' - v')\, dr\Big]\, ds \right\| \leqslant \int_0^\tau | G_t(t, s)|\, h(s)\, ds \, \| u - v \|_F \\ &\leqslant \lambda \| u - v \|_F \end{aligned}$$

we have that L is a contraction. Again the argument is independent of the choice of β so (1), (2) with $\alpha = \Theta$ is uniquely solvable. But this means $U(\tau)$ in Lemma 1 is invertible, hence the conclusion.

Corollary. *If A and B are constant and $3\tau \| A \| + 2\tau^2 \| B \| < 6$,* (1), (2) *is uniquely solvable.*

THEOREM 5. *Assume* A, A' *and* B *are continuous with* $2\int_0^t s\,\|A(s)+(t-s)[B(s)-A'(s)]\|\,ds \leqslant t$ *on* $[0, \tau]$. *Then* (1), (2) *is uniquely solvable.*

Proof. For the operator U of Lemma 1 we have $\|U(t)-tI\| < t$ on some interval $0 < t < \epsilon$. Let $(0, \lambda)$ denote the maximal interval on which the strict inequality holds. Then $\|U(t)\| < 2t$ on $(0, \lambda)$. If $\lambda \leqslant \tau$, we have

$$\lambda = \|U(\lambda) - \lambda I\| = \left\|\int_0^\lambda [A + (\lambda - s)(B - A')]\,U\,ds\right\|$$

$$< 2\int_0^\lambda s\,\|A + (\lambda - s)(B - A')\|\,ds \leqslant \lambda,$$

a contradiction. Hence $\lambda > \tau$ so that $\|\tau^{-1}U(\tau) - I\| < 1$ and $U(\tau)$ is invertible.

Unfortunately the preceding conditions are monotone in τ. When $X = [X] =$ real numbers, the solution U of (3) is simply the scalar solution u to (1) satisfying $u(0) = 0$, $u'(0) = 1$. Since the zeros of u can only cluster at $+\infty$, (1), (2) is solvable on $[0, \tau]$ for almost every $\tau > 0$. In the general case one can construct an example of a system (1) such that the solution U of (3) has singularities (real numbers $t \geqslant 0$ for which $U(t)x = \Theta$ for some $\Theta \neq x \in X$) with a finite cluster point. However, it seems likely that in many cases $U(t)$ might again become invertible for t sufficiently large. If A is continuously differentiable and $K(t, s) = A(s) + (t - s)[B(s) - A'(s)]$ for $0 \leqslant s \leqslant t$, U solves the equation $U(t) = tI + \int_0^t K(t, s)\,U(s)\,ds$. Using successive iterations we can represent the solution formally as

$$U(t) = tI + \int_0^t sH(t, s)\,ds \qquad \text{where} \qquad H(t, s) = \sum_{n=0}^{\infty} K_n(t, s)$$

and

$$K_n(t, s) = \begin{cases} K(t, s) & n = 0 \\ \int_0^t K(t, u)\,K_{n-1}(u, s)\,du, & n \geqslant 1 \end{cases}$$

For $\tau > 0$ fixed, the series $H(t, s)$ converges in norm uniformly on $0 \leqslant s \leqslant t \leqslant \tau$. If

$$\left\|\int_0^\tau sH(\tau, s)\,ds\right\| < \tau \qquad \text{for some} \qquad \tau > 0, \qquad \text{then} \tag{6}$$

$\|\tau^{-1}U(\tau) - I\| < 1$ and $U(\tau)$ is invertible. For certain specified A and B it may be possible to obtain estimates which imply (6) for certain large values of τ and hence (1), (2) would be solvable on such intervals.

5. Hilbert Space

In this section X will be a real Hilbert space with the inner product of two elements $x, y \in X$ denoted by $x \cdot y$. For an operator $T \in [X]$ we write T^* for the adjoint of T, T^0 for $\frac{1}{2}(T + T^*)$ and $T \geqslant 0$ in case $x \cdot Tx \geqslant 0$ for all $x \in X$.

The following condition is given in [3] as a sufficient condition that (1) be disconjugate on $[0, \tau]$. A minor modification of the original proof shows that the condition also implies that (1), (2) is solvable.

Theorem 6. *In order that* (1), (2) *be uniquely solvable on* $[0, \tau]$, *it is sufficient that there exist a continuously differentiable operator* $K(t)$ *such that* $B + K' \geqslant (\frac{1}{2}A + K^0)(\frac{1}{2}A^* + K^0)$ on $[0, \tau]$.

Proof. Again we will show that $U(\tau)$ is invertible.

Let y be any solution of (1) such that $y(0) = \Theta$; i.e. $y(t) = U(t)x$ for some $x \in X$. Then for $0 < t \leqslant \tau$,

$$\begin{aligned} &y(t) \cdot y'(t) + y(t) \cdot K(t)\, y(t) \\ &\quad = \int_0^t [(y \cdot y') + (y \cdot Ky)']\, ds \\ &\quad = \int_0^t [y' \cdot y' + y \cdot Ay' + y \cdot By + 2y' \cdot K^0 y + y \cdot K'y]\, ds \\ &\quad = \int_0^t [\| y' + (\tfrac{1}{2}A^* + K^0)\, y \|^2 + y \cdot Ly]\, ds \end{aligned} \tag{7}$$

where $L = B + K' - (\frac{1}{2}A + K^0)(\frac{1}{2}A^* + K^0) \geqslant 0$. If $y(\lambda) = \Theta$ for any $0 < \lambda \leqslant \tau$, the integrand on the right must be identically zero on $[0, \lambda]$ so y is a solution to the first order *IVP* $y' = -(\frac{1}{2}A^* + K^0)y$, $y(0) = \Theta$. By uniqueness we must have $y \equiv \Theta$ and it follows that $U(t)$ is one-one on $(0, \tau]$.

Now U is invertible on some open interval to the right of zero so if U fails to remain invertible on $[0, \tau]$ there is a first point λ, $0 < \lambda \leqslant \tau$ such that $\lim_{t \to \lambda^-} \| U^{-1}(t) \| = +\infty$. Since U is already one–one at λ, there must exist a sequence $\{x_n\}$ in X with $\| x_n \| = 1$ and $\lim_{n \to \infty} U(\lambda)x_n = \Theta$. But this is incompatible with (7). For the initial conditions $U(0) = \Theta$, $U'(0) = I$ require that $\| Z(t) - I \| \leqslant \frac{1}{2}$ on some interval $0 \leqslant t \leqslant \in < \lambda$ where $Z = U' + (\frac{1}{2}A^* + K^0)U$. Consequently Z^{-1} exists and is bounded on $[0, \in]$, say $\| Z^{-1}(t) \| \leqslant M$. It follows that $\| Z(t)\, x \| \geqslant (1/M)$ on $[0, \in]$ for $x \in X$ with $\| x \| = 1$. Putting $y_n(t) = U(t)x_n$ into (7) we get

$$0 \geqslant \lim_{n \to \infty} \int_0^\lambda \| Z(t)\, x_n \|^2\, ds \geqslant \frac{\epsilon}{M^2}.$$

From this contradiction we must have $U(\tau)$ invertible.

For $A = \Phi$, the inequality in Theorem 6 becomes $K' + B \geqslant K^{0}$ so (1), (2) is uniquely solvable on any interval $[0, \tau]$ on which the "Riccati" equation $K' + B = K^{0^2}$ has a solution.

COROLLARY. *Let A, A' and B be continuous on $[0, \tau]$ into $[X]$. Then (1), (2) is uniquely solvable if any of the following conditions hold on $[0, \tau]$.*

(i) $A = A^*$ *and* $B \geqslant \frac{1}{2}A'$

(ii) $A = -A^*$ *and* $B + \frac{1}{4}A^2 \geqslant \frac{1}{2}A'$

(iii) $B \geqslant \frac{1}{4}AA^*$

(iv) $B \geqslant A' + \frac{1}{4}A^*A$

Proof. In (i) and (ii) take $K = -\frac{1}{2}A$. Taking $K = \Theta$ and $-A$ gives (iii) and (iv).

From condition (i) we observe that for given B the system (1) can always be overdamped by taking $A(t) = g(t)I$ where $g(t)$ is any scalar function such that $g'(t) \leqslant -2\,\|B(t)\|$ on $[0, \tau]$.

THEOREM 7. *Let X be a separable Hilbert space with $\{e_n\}$ a complete orthonormal set in X. Let $b_{ij}(t) = B(t)\, e_j \cdot e_i$, $a_{ij}(t) = A(t)\, e_j \cdot e_i$. If $a_{ij}(t) = 0$ and $b_{ij}(t) \leqslant 0$ on $[0, \tau]$ for $i \neq j$ while $\sum_{j=1}^{\infty} b_{ij}(t) \geqslant 0$ on $[0, \tau]$ for each $i \geqslant 1$. Then (1) is disconjugate on $[0, \tau]$. If A and B are also completely continuous operators, (1), (2) is uniquely solvable on $[0, \tau]$.*

Proof. It is convenient to think of A and B as infinite square matrices and y as an infinite column vector.

Suppose there is a nontrivial solution $y(t)$ to (1) with $y(t_1) = y(t_2) = \Theta$ for some $0 \leqslant t_1 < t_2 \leqslant \tau$. Taking y or $-y$, there is some component $y_k = y \cdot e_k$ of y and some $t_0 \in (t_1, t_2)$ such that

$$y_k(t_0) = \sup\{y_j(t) : t_1 \leqslant t \leqslant t_2, j \geqslant 1\} > 0.$$

Then $y_k'(t_0) = 0$ so

$$y_k''(t_0) = a_{kk}(t_0)\, y_k'(t_0) + \sum_{j=1}^{\infty} b_{kj}(t_0)\, y_j(t_0)$$

$$\geqslant \sum_{y_j(t_0)>0} b_{kj}(t_0)\, y_j(t_0) \geqslant \Big(\sum_{j=1}^{\infty} b_{kj}(t_0)\Big)\, y_k(t_0) \geqslant 0$$

which contradicts y_k having a positive interior maximum. Hence (1) is disconjugate on $[0, \tau]$. It follows from Lemma 2 that (1), (2) is uniquely solvable with the additional assumption of complete continuity on A and B.

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